Behind the Scenes of Chronological CDCL

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Backtracking in Out-of-Order Trail

τ	0	1	**2**	3	**4**
I	1	2	**3**	4	**5**
δ	0	1	**0**	2	**1**

τ	0	1	**2**	3
I	1	2	**3**	5
δ	0	1	**0**	1

τ	0	1
I	1	3
δ	0	0

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	**5**	30	47	15	18	**6**	-7	-8	45	**9**	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

decision literal

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

$\text{block}(I, 4)$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

$\text{slice}(I, 4)$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	**4**	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

$I_{\leqslant 4}$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16	✗
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6	

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I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

✗

conflicting { -47, -17, -44 }

conflict level 6

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16	✗
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6	
conflicting		{			-47,										-17,	-44	}	
learned		{		-30,	-47,		-18,							23			}	

jump level 4

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6
conflicting		{			-47,										-17,	-44	}
learned		{		-30,	-47,		-18,							23			}

backtrack level 5

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	**23**
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	**4**

out of order

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14
I	$\cdots$	4	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	23
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	4

block$(I, 4)$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14
I	$\cdots$	4	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	**23**
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	4

$\text{slice}(I, 4)$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14
I	$\cdots$	**4**	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	**23**
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	4

$I_{\leqslant 4}$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I	$\cdots$	18	6	-7	-8	45	23	-38	-9	10	-11	13	16	-17	-25	42	12	-41
δ	$\cdots$	4	5	5	5	5	4	4	5	6	7	5	4	4	4	4	5	5

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I	$\cdots$	18	6	-7	-8	45	23	-38	-9	10	-11	13	16	-17	-25	42	12	-41
δ	$\cdots$	4	5	5	5	5	4	4	5	6	7	5	4	4	4	4	5	5
																		✗
conflicting		{												17,		-42,	-12	}

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I	$\cdots$	18	6	-7	-8	45	23	-38	-9	10	-11	13	16	-17	-25	42	12	-41
δ	$\cdots$	4	5	5	5	5	4	4	5	6	7	5	4	4	4	4	5	5

conflicting $\{$ 17, -42, -12 $\}$

backtrack level 4

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11
I	$\cdots$	18	23	-38	16	-17	-25	42	-12
δ	$\cdots$	4	4	4	4	4	4	4	4

CDCL Invariants

Trail:	The assignment trail contains neither complementary pairs of literals nor duplicates.
ConflictLower:	The assignment trail preceding the current decision level does not falsify the formula.
Propagation:	On every decision level preceding the current decision level all unit clauses are propagated until completion.
LevelOrder:	The literals are ordered on the assignment trail in ascending order with respect to their decision level.
ConflictingClause:	At decision levels greater than zero the conflicting clause contains at least two literals with the current decision level.

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Invariants

Trail: The assignment trail contains neither complementary pairs of literals nor duplicates.

ConflictLower: The assignment trail preceding the current decision level does not falsify the formula.

(1): $\forall k, \ell \in \mathsf{decs}(I) \,.\, \tau(I, k) < \tau(I, \ell) \implies \delta(k) < \delta(\ell)$

(2): $\delta(\mathsf{decs}(I)) = \{1, \ldots, \delta(I)\}$

(3): $\forall n \in \mathbb{N} \,.\, F \wedge \mathsf{decs}_{\leqslant n}(I) \models I_{\leqslant n}$

Input: formula F, set of variables V, trail I, decision level function δ
Output: SAT iff F is satisfiable, UNSAT otherwise

Search(F)

$V := V(F)$; $I := \varepsilon$; $\delta := \infty$
while there are unassigned variables in V **do**
  $C :=$ *Propagate*(F, I, δ)
  if $C \neq \bot$ **then**
    $c := \delta(C)$
    if $c = 0$ **then return** UNSAT
    Analyze(F, I, C, c)
  else
    Decide(I, δ)
return SAT

Propagate(F, I, δ)

while some $C \in F$ is unit $\{\ell\}$ under I **do**
  $I := I\ell$; $\delta(\ell) := \delta(C \setminus \{\ell\})$
  for all clauses $D \in F$ containing $\neg\ell$ **do**
    if $I(D) = \bot$ **then return** D
return $\bot$

Analyze(F, I, C, c)

if C contains exactly one literal at decision level c **then**
  $\ell :=$ literal in C at decision level c
  $j := \delta(C \setminus \{\ell\})$
else
  $D :=$ *Learn*(I, C)
  $F := F \wedge D$
  $\ell :=$ literal in D at decision level c
  $j := \delta(D \setminus \{\ell\})$
pick $b \in [j,\ c-1]$
for all literals $k \in I$ with decision level $> b$ **do**
  assign k decision level ∞
  remove k from I
$I := I\ell$
assign ℓ decision level j

Calculus

True: $(F,\ I,\ \delta) \rightsquigarrow_{\text{True}}$ SAT if $F|_I = \top$

False: $(F,\ I,\ \delta) \rightsquigarrow_{\text{False}}$ UNSAT if exists $C \in F$ with $C|_I = \bot$ and $\delta(C) = 0$

Unit: $(F,\ I,\ \delta) \rightsquigarrow_{\text{Unit}} (F,\ I\ell,\ \delta[\ell \mapsto a])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and exists $C \in F$ with $\{\ell\} = C|_I$ and $a = \delta(C \setminus \{\ell\})$

Jump: $(F,\ I,\ \delta) \rightsquigarrow_{\text{Jump}} (F \wedge D,\ PK\ell,\ \delta[L \mapsto \infty][\ell \mapsto j])$ if exists $C \in F$ with $PQ = I$ and $C|_I = \bot$ such that $c = \delta(C) = \delta(D) > 0$ and $\ell \in D$ and $\ell|_Q = \bot$ and $F \models D$ and $j = \delta(D \setminus \{\ell\})$ and $b = \delta(P)$ and $j \leqslant b < c$ and $K = Q_{\leqslant b}$ and $L = Q_{>b}$

Decide: $(F,\ I,\ \delta) \rightsquigarrow_{\text{Decide}} (F,\ I\ell,\ \delta[\ell \mapsto d])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and $\text{units}(F|_I) = \emptyset$ and $V(\ell) \in V$ and $\delta(\ell) = \infty$ and $d = \delta(I) + 1$

Calculus

True: $(F,\ I,\ \delta) \rightsquigarrow_{\text{True}}$ SAT if $F|_I = \top$

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Decide: $(F,\ I,\ \delta) \rightsquigarrow_{\text{Decide}} (F,\ I\ell,\ \delta[\ell \mapsto d])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and $\text{units}(F|_I) = \emptyset$ and $V(\ell) \in V$ and $\delta(\ell) = \infty$ and $d = \delta(I) + 1$

Calculus

True: $(F,\ I,\ \delta) \leadsto_{\text{True}}$ SAT if $F|_I = \top$

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Jump: $(F,\ I,\ \delta) \leadsto_{\text{Jump}} (F \wedge D,\ PK\ell,\ \delta[L \mapsto \infty][\ell \mapsto j])$ if exists $C \in F$ with $PQ = I$ and $C|_I = \bot$ such that $c = \delta(C) = \delta(D) > 0$ and $\ell \in D$ and $\ell|_Q = \bot$ and $F \models D$ and $j = \delta(D \setminus \{\ell\})$ and $b = \delta(P)$ and $b = j$ and $K = Q_{\leqslant b}$ and $L = Q_{>b}$

Decide: $(F,\ I,\ \delta) \leadsto_{\text{Decide}} (F,\ I\ell,\ \delta[\ell \mapsto d])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and $\text{units}(F|_I) = \emptyset$ and $V(\ell) \in V$ and $\delta(\ell) = \infty$ and $d = \delta(I) + 1$