Behind the Scenes of Chronological CDCL

Sibylle Möhle and Armin Biere

Institute for Formal Models and Verification
LIT Secure and Correct Systems Lab

JOHANNES KEPLER
UNIVERSITY LINZ

June 26, 2020

Backtracking in Out-of-Order Trail

τ	0	1	**2**	3	**4**
I	1	2	**3**	4	**5**
δ	0	1	**0**	2	**1**

τ	0	1	**2**	3
I	1	2	**3**	5
δ	0	1	**0**	1

τ	0	1
I	1	3
δ	0	0

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	**5**	30	47	15	18	**6**	-7	-8	45	**9**	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

decision literal

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

$$\mathsf{block}(I, 4)$$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

$\text{slice}(I, 4)$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	**4**	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

$$I_{\leqslant 4}$$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6

✗

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16	✗
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6	
conflicting		{			-47,										-17,	-44	}	

conflict level 6

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16	✗
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6	
conflicting		{			-47,										-17,	-44	}	
learned		{		-30,	-47,		-18,							23			}	

jump level 4

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	9	38	-23	17	44	-16
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	6	6	6	6	6	6
conflicting		$\{$			-47,										-17,	-44	$\}$
learned		$\{$		-30,	-47,		-18,							23			$\}$

✗

backtrack level 5

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45	**23**
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	**4**

out of order

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14
I	$\cdots$	4	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	23
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	4

block$(I, 4)$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14
I	$\cdots$	4	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	**23**
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	4

slice$(I, 4)$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13
I	$\cdots$	4	5	30	47	15	18	6	-7	-8	45
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14
I	$\cdots$	**4**	**5**	**30**	**47**	**15**	**18**	6	-7	-8	45	**23**
δ	$\cdots$	3	4	4	4	4	4	5	5	5	5	4

$I_{\leqslant 4}$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I	$\cdots$	18	6	-7	-8	45	23	-38	-9	10	-11	13	16	-17	-25	42	12	-41
δ	$\cdots$	4	5	5	5	5	4	4	5	6	7	5	4	4	4	4	5	5

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I	$\cdots$	18	6	-7	-8	45	23	-38	-9	10	-11	13	16	-17	-25	42	12	-41
δ	$\cdots$	4	5	5	5	5	4	4	5	6	7	5	4	4	4	4	5	5
																		✗
conflicting		$\{$												17,		-42,	-12	$\}$

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I	$\cdots$	18	6	-7	-8	45	23	-38	-9	10	-11	13	16	-17	-25	42	12	-41
δ	$\cdots$	4	5	5	5	5	4	4	5	6	7	5	4	4	4	4	5	5

✗

conflicting { 17, -42, -12 }

backtrack level 4

Combining CDCL with Chronological Backtracking

τ	$\cdots$	4	5	6	7	8	9	10	11
I	$\cdots$	18	23	-38	16	-17	-25	42	-12
δ	$\cdots$	4	4	4	4	4	4	4	4

CDCL Invariants

Trail:	The assignment trail contains neither complementary pairs of literals nor duplicates.
ConflictLower:	The assignment trail preceding the current decision level does not falsify the formula.
Propagation:	On every decision level preceding the current decision level all unit clauses are propagated until completion.
LevelOrder:	The literals are ordered on the assignment trail in ascending order with respect to their decision level.
ConflictingClause:	At decision levels greater than zero the conflicting clause contains at least two literals with the current decision level.

CDCL Invariants

Trail:	The assignment trail contains neither complementary pairs of literals nor duplicates.
ConflictLower:	The assignment trail preceding the current decision level does not falsify the formula.
Propagation:	On every decision level preceding the current decision level all unit clauses are propagated until completion.
LevelOrder:	The literals are ordered on the assignment trail in ascending order with respect to their decision level.
ConflictingClause:	At decision levels greater than zero the conflicting clause contains at least two literals with the current decision level.

Invariants

Trail: The assignment trail contains neither complementary pairs of literals nor duplicates.

ConflictLower: The assignment trail preceding the current decision level does not falsify the formula.

(1): $\forall k, \ell \in \text{decs}(I) \;.\; \tau(I, k) < \tau(I, \ell) \implies \delta(k) < \delta(\ell)$

(2): $\delta(\text{decs}(I)) = \{1, \ldots, \delta(I)\}$

(3): $\forall n \in \mathbb{N} \;.\; F \wedge \text{decs}_{\leqslant n}(I) \models I_{\leqslant n}$

Input: formula F, set of variables V, trail I, decision level function δ
Output: SAT iff F is satisfiable, UNSAT otherwise

Search(F)

$V := V(F)$; $I := \varepsilon$; $\delta := \infty$
while there are unassigned variables in V **do**
  $C :=$ *Propagate*(F, I, δ)
  if $C \neq \bot$ **then**
    $c := \delta(C)$
    if $c = 0$ **then return** UNSAT
    Analyze(F, I, C, c)
  else
    Decide(I, δ)
return SAT

Propagate(F, I, δ)

while some $C \in F$ is unit $\{\ell\}$ under I **do**
  $I := I\ell$; $\delta(\ell) := \delta(C \setminus \{\ell\})$
  for all clauses $D \in F$ containing $\neg\ell$ **do**
    if $I(D) = \bot$ **then return** D
return $\bot$

Analyze(F, I, C, c)

if C contains exactly one literal at decision level c **then**
  $\ell :=$ literal in C at decision level c
  $j := \delta(C \setminus \{\ell\})$
else
  $D :=$ *Learn*(I, C)
  $F := F \wedge D$
  $\ell :=$ literal in D at decision level c
  $j := \delta(D \setminus \{\ell\})$
pick $b \in [j,\ c-1]$
for all literals $k \in I$ with decision level $> b$ **do**
  assign k decision level ∞
  remove k from I
$I := I\ell$
assign ℓ decision level j

Calculus

True: $(F,\ I,\ \delta) \rightsquigarrow_{\text{True}}$ SAT if $F|_I = \top$

False: $(F,\ I,\ \delta) \rightsquigarrow_{\text{False}}$ UNSAT if exists $C \in F$ with $C|_I = \bot$ and $\delta(C) = 0$

Unit: $(F,\ I,\ \delta) \rightsquigarrow_{\text{Unit}} (F,\ I\ell,\ \delta[\ell \mapsto a])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and exists $C \in F$ with $\{\ell\} = C|_I$ and $a = \delta(C \setminus \{\ell\})$

Jump: $(F,\ I,\ \delta) \rightsquigarrow_{\text{Jump}} (F \wedge D,\ PK\ell,\ \delta[L \mapsto \infty][\ell \mapsto j])$ if exists $C \in F$ with $PQ = I$ and $C|_I = \bot$ such that $c = \delta(C) = \delta(D) > 0$ and $\ell \in D$ and $\ell|_Q = \bot$ and $F \models D$ and $j = \delta(D \setminus \{\ell\})$ and $b = \delta(P)$ and $j \leqslant b < c$ and $K = Q_{\leqslant b}$ and $L = Q_{>b}$

Decide: $(F,\ I,\ \delta) \rightsquigarrow_{\text{Decide}} (F,\ I\ell,\ \delta[\ell \mapsto d])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and $\text{units}(F|_I) = \emptyset$ and $V(\ell) \in V$ and $\delta(\ell) = \infty$ and $d = \delta(I) + 1$

Calculus

True: $(F,\ I,\ \delta) \leadsto_{\text{True}}$ SAT if $F|_I = \top$

False: $(F,\ I,\ \delta) \leadsto_{\text{False}}$ UNSAT if exists $C \in F$ with $C|_I = \bot$ and $\delta(C) = 0$

Unit: $(F,\ I,\ \delta) \leadsto_{\text{Unit}} (F,\ I\ell,\ \delta[\ell \mapsto a])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and exists $C \in F$ with $\{\ell\} = C|_I$ and $a = \delta(C \setminus \{\ell\})$

Jump: $(F,\ I,\ \delta) \leadsto_{\text{Jump}} (F \wedge D,\ PK\ell,\ \delta[L \mapsto \infty][\ell \mapsto j])$ if exists $C \in F$ with $PQ = I$ and $C|_I = \bot$ such that $c = \delta(C) = \delta(D) > 0$ and $\ell \in D$ and $\ell|_Q = \bot$ and $F \models D$ and $j = \delta(D \setminus \{\ell\})$ and $b = \delta(P)$ and $b = c - 1$ and $K = Q_{\leqslant b}$ and $L = Q_{>b}$

Decide: $(F,\ I,\ \delta) \leadsto_{\text{Decide}} (F,\ I\ell,\ \delta[\ell \mapsto d])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and $\text{units}(F|_I) = \emptyset$ and $V(\ell) \in V$ and $\delta(\ell) = \infty$ and $d = \delta(I) + 1$

Calculus

True: $(F,\ I,\ \delta) \leadsto_{\mathsf{True}}$ SAT if $F|_I = \top$

False: $(F,\ I,\ \delta) \leadsto_{\mathsf{False}}$ UNSAT if exists $C \in F$ with $C|_I = \bot$ and $\delta(C) = 0$

Unit: $(F,\ I,\ \delta) \leadsto_{\mathsf{Unit}} (F,\ I\ell,\ \delta[\ell \mapsto a])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and exists $C \in F$ with $\{\ell\} = C|_I$ and $a = \delta(C \setminus \{\ell\})$

Jump: $(F,\ I,\ \delta) \leadsto_{\mathsf{Jump}} (F \wedge D,\ PK\ell,\ \delta[L \mapsto \infty][\ell \mapsto j])$ if exists $C \in F$ with $PQ = I$ and $C|_I = \bot$ such that $c = \delta(C) = \delta(D) > 0$ and $\ell \in D$ and $\ell|_Q = \bot$ and $F \models D$ and $j = \delta(D \setminus \{\ell\})$ and $b = \delta(P)$ and $b = j$ and $K = Q_{\leqslant b}$ and $L = Q_{>b}$

Decide: $(F,\ I,\ \delta) \leadsto_{\mathsf{Decide}} (F,\ I\ell,\ \delta[\ell \mapsto d])$ if $F|_I \neq \top$ and $\bot \notin F|_I$ and $\mathsf{units}(F|_I) = \emptyset$ and $V(\ell) \in V$ and $\delta(\ell) = \infty$ and $d = \delta(I) + 1$